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LETTER TO THE EDITOR

A vector Lagrangian for the electromagnetic field

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Abstract. A variational principle for Maxwell's equations in which the variables are the electromagnetic field strengths is formulated covariantly; the Lagrangian density is a 4-vector. Conserved quantities associated with translations, Lorentz transformations and duality rotations are determined.

It has been pointed out by Anderson and Arthurs (1978) and Rosen (1980) that Maxwell's equations for the electromagnetic field can be derived from a variational principle in which the independent variables are the electric and magnetic fields (and not, as usual, the potentials). In this letter we present a relativistically covariant formulation of this variational principle. This has the interesting feature that the Lagrangian is not a Lorentz scalar but a 4-vector. We examine the invariances of this Lagrangian, and show that its translational invariance provides an explanation for a tensor of conserved quantities discovered by Lipkin (1964). This formulation also makes it possible to treat duality rotations as a symmetry of the Lagrangian in the usual way.

The variational principle of Anderson and Arthurs (1978) and Rosen (1980) is that the integral

$$S_0=\int\left(\boldsymbol{B}\cdot\frac{\partial \boldsymbol{E}}{\partial t}-\boldsymbol{E}\cdot\frac{\partial \boldsymbol{B}}{\partial t}-\boldsymbol{E}\cdot(\nabla\times\boldsymbol{E})-\boldsymbol{B}\cdot(\nabla\times\boldsymbol{B})+2\boldsymbol{j}\cdot\boldsymbol{B}\right)\mathrm{d}^3\boldsymbol{r}\,\mathrm{d}t \tag{1}$$

should be stationary under variations of $\boldsymbol{E}$ and $\boldsymbol{B}$, the current density $\boldsymbol{j}$ being given. The Euler-Lagrange equations for this problem are the partial set of Maxwell equations

$$\nabla\times\boldsymbol{E}=-\partial\boldsymbol{B}/\partial t, \qquad \nabla\times\boldsymbol{B}=\boldsymbol{j}+\partial\boldsymbol{E}/\partial t. \tag{2}$$

Let $\mathscr{L}_0$ be the integrand of (1). This is the time component of a 4-vector

$$\mathscr{L}_\alpha={}^*F^{\mu\nu}\partial_\nu F_{\mu\alpha}-F^{\mu\nu}\partial_\nu{}^*F_{\mu\alpha}-2{}^*F_{\alpha\mu}j^\mu \tag{3}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor ($F_{0i}=E_i$, $F_{ij}=-\varepsilon_{ijk}B_k$), ${}^*F_{\mu\nu}=\varepsilon_{\mu\nu\rho\sigma}F$ is its dual (${}^*F_{0i}=-B_i$, ${}^*F_{ij}=-\varepsilon_{ijk}E_k$), and $j^\mu=(\rho,\boldsymbol{j})$ is the current 4-vector. The four simultaneous variational principles that

$$S_\alpha=\int\mathscr{L}_\alpha\,\mathrm{d}^4x \tag{4}$$

should be stationary under variations of $F_{\mu\nu}$, j^μ being fixed, lead to Euler-Lagrange equations

$$\partial_\rho(\partial\mathscr{L}_\alpha/\partial(\partial_\rho F^{\mu\nu}))-\partial\mathscr{L}_\alpha/\partial F^{\mu\nu}-(\mu\leftrightarrow\nu)$$
$$=2\varepsilon_{\mu\nu\alpha\beta}(j^\beta-\partial_\rho F^{\rho\beta})+2(g_{\alpha\mu}g_{\beta\nu}-g_{\alpha\nu}g_{\beta\mu})\partial_\rho{}^*F^{\rho\beta}=0. \quad (5)$$

These are equivalent to the full set of Maxwell's equations

$$\partial_\mu F^{\mu\nu}=j^\nu, \qquad \partial_\mu{}^*F^{\mu\nu}=0. \quad (6)$$

Note that when these field equations are satisfied, the value of the Lagrangian density becomes

$$\mathscr{L}_\alpha=-{}^*F_{\alpha\mu}j^\mu. \quad (7)$$

The space part of the 4-vector Lagrangian density $\mathscr{L}_\alpha$ is

$$\boldsymbol{\mathscr{L}}=\boldsymbol{E}\times(\boldsymbol{\nabla}\times\boldsymbol{B})-\boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{E})+(\boldsymbol{\nabla}\cdot\boldsymbol{E})\boldsymbol{B}$$
$$-(\boldsymbol{\nabla}\cdot\boldsymbol{B})\boldsymbol{E}-\boldsymbol{E}\times\partial\boldsymbol{E}/\partial t-\boldsymbol{B}\times\partial\boldsymbol{B}/\partial t+2\boldsymbol{j}\times\boldsymbol{E}-2\rho\boldsymbol{B}. \quad (8)$$

The variational principle for the integral of $\boldsymbol{\mathscr{L}}$ yield Euler–Lagrange equations

$$\varepsilon_{ijk}(\partial E_k/\partial t+j_k)+\partial_j B_j-\partial_i B_j+\boldsymbol{\nabla}\cdot\boldsymbol{B}\delta_{ij}=0,$$
$$\varepsilon_{ijk}\,\partial B_k/\partial t+\partial_i E_j-\partial_j E_i-(\boldsymbol{\nabla}\cdot\boldsymbol{E}-\rho)\delta_{ij}=0,$$

whose symmetric and antisymmetric parts give the usual four Maxwell's equations.

It is probably impossible to make the Lagrangian (3) yield the equations of motion for charged matter by treating the source variables as independent degrees of freedom and adding a kinetic term.

In the absence of sources ($j^\mu=0$) the action (4) is invariant under spacetime translations. The argument which normally leads to the divergence-free energy-momentum tensor T_μ^ν yields in this case a third-rank tensor

$$U_{\alpha\mu}{}^\nu=(\partial\mathscr{L}_\alpha/\partial(\partial_\nu F^{\rho\sigma}))\partial_\mu F^{\rho\sigma}-\mathscr{L}_\alpha\delta_\mu^\nu$$
$$={}^*F^{\rho\nu}\partial_\mu F_{\rho\alpha}-F^{\rho\nu}\partial_\mu{}^*F_{\rho\alpha}-({}^*F^{\rho\sigma}\partial_\sigma F^{\rho\alpha}-F^{\rho\sigma}\partial_\sigma{}^*F_{\rho\alpha})\delta_\mu^\nu \quad (9)$$

which satisfies $\partial_\nu U_{\alpha\mu}{}^\nu=0$. It is related to Lipkin's tensor $Z_{\alpha\mu\nu}$ by

$$Z_{\alpha\mu\nu}=-\tfrac{1}{4}(U_{\alpha\mu\nu}+\mathscr{L}_\alpha g_{\mu\nu})+\tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\partial^\rho F^{\sigma\tau}F_{\tau\alpha}+(\alpha\leftrightarrow\mu). \quad (10)$$

Since $\mathscr{L}_\alpha$ vanishes as a consequence of the source-free field equations, and the third term, a four-dimensional curl, is automatically divergence-free, Lipkin's tensor is essentially the symmetric part of $U_{\alpha\mu}{}^\nu$. The equation $\partial_\nu Z_{\alpha\mu}{}^\nu=0$ expresses the conservation of the scalar, the 3-vector and the symmetric traceless 3-tensor whose densities are

$$Z=\boldsymbol{E}\cdot(\boldsymbol{\nabla}\times\boldsymbol{E})+\boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}), \quad (11)$$

$$\boldsymbol{Z}=\boldsymbol{E}\times(\boldsymbol{\nabla}\times\boldsymbol{B})-\boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{E}) \quad (12)$$

and the traceless part of

$$\underline{\underline{Z}}=\boldsymbol{E}\vee(\boldsymbol{\nabla}\times\boldsymbol{E})+\boldsymbol{B}\vee(\boldsymbol{\nabla}\times\boldsymbol{B}) \quad (13)$$

where $\boldsymbol{u}\vee\boldsymbol{v}$ denotes the symmetric tensor $u_i v_j+u_j v_i$.

The antisymmetric part of $U_{\alpha\mu}{}^\nu$ can, using the field equations, be put in the form

$$X_{\alpha\mu}{}^\nu=U_{\alpha\mu}{}^\nu-U_{\mu\alpha}{}^\nu={}^*F^{\nu\rho}\partial_\rho F_{\alpha\mu}-F^{\nu\rho}\partial_\rho{}^*F_{\alpha\mu}. \quad (14)$$

The corresponding conserved quantities are the vectors with densities

$$\boldsymbol{X}=(\boldsymbol{E}\cdot\nabla)\boldsymbol{E}+(\boldsymbol{B}\cdot\nabla)\boldsymbol{B} \tag{15}$$

and

$$\boldsymbol{Y}=(\boldsymbol{E}\cdot\nabla)\boldsymbol{B}-(\boldsymbol{B}\cdot\nabla)\boldsymbol{E}. \tag{16}$$

Unlike the symmetric part (4), the antisymmetric tensor $X_{\alpha\mu}{}^{\nu}$ consists of derivatives of the energy-momentum tensor T^{ν}_{μ}:

$$X^{\alpha\mu\nu}=-\partial_{\rho}(\varepsilon^{\rho\sigma\mu\nu}T_{\sigma}{}^{\alpha}-\varepsilon^{\rho\alpha\sigma\nu}T_{\sigma}{}^{\mu}+\varepsilon^{\rho\alpha\mu\sigma}T_{\sigma}{}^{\nu}). \tag{17}$$

Lorentz transformations give rise to a set of currents forming the tensor

$$\begin{aligned} j_{\alpha\kappa\lambda}{}^{\nu}&=\left(\frac{\partial\mathscr{L}_{\alpha}}{\partial(\partial_{\nu}F^{\kappa\rho})}\right)F_{\lambda}{}^{\rho}+\left(\frac{\partial\mathscr{L}_{\alpha}}{\partial(\partial_{\nu}F^{\rho\kappa})}\right)F^{\rho}{}_{\lambda}+x_{\kappa}T_{\alpha\lambda}{}^{\nu}-(\kappa\leftrightarrow\lambda)\\ &={}^{*}F_{\kappa}{}^{\nu}F_{\lambda\alpha}+{}^{*}F^{\nu\rho}F_{\lambda\rho}g_{\alpha\kappa}+\varepsilon_{\alpha\kappa\rho\sigma}F^{\nu\rho}F_{\lambda}{}^{\sigma}+x_{\kappa}T_{\alpha\lambda}{}^{\nu}-(\kappa\leftrightarrow\lambda). \end{aligned} \tag{18}$$

The vector nature of the Lagrangian means that these currents do not describe the flow of conserved quantities; instead of a continuity equation, they satisfy

$$\partial_{\nu}j_{\alpha\kappa\lambda}{}^{\nu}=g_{\alpha\kappa}\mathscr{L}_{\lambda}-g_{\alpha\lambda}\mathscr{L}_{\kappa}. \tag{19}$$

However, for any constant 4-vector t^{α} the component $t^{\alpha}\mathscr{L}_{\alpha}$ is invariant under Lorentz transformations belonging to the little group of t^{α}. This means that for every 4-vector t^{α} and antisymmetric tensor $\omega_{\alpha\beta}$ satisfying $\omega_{\alpha\beta}t^{\beta}=0$ there is a conserving current

$$j^{\nu}=t^{\alpha}\omega^{\kappa\lambda}j_{\alpha\kappa\lambda}{}^{\nu}. \tag{20}$$

In the case of the timelike 4-vector $t^{\alpha}=(1,\boldsymbol{0})$ we obtain the components of a conserved 3-vector associated with the rotational invariance of $\mathscr{L}_0$, the integrand of (1). The density $J_i=\frac{1}{2}\varepsilon_{ijk}j_{0jk}{}^{0}$ of this angular momentum-like vector is

$$\boldsymbol{J}=2\boldsymbol{E}\times\boldsymbol{B}+\boldsymbol{r}\times(\boldsymbol{Y}+\boldsymbol{Z}) \tag{21}$$

where $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are the vectors of (12) and (16).

With the field strengths as basic variables it is possible to implement the duality rotations

$$F_{\mu\nu}\rightarrow F_{\mu\nu}\cos\theta+{}^{*}F_{\mu\nu}\sin\theta \tag{22}$$

i.e.

$$\boldsymbol{E}+\mathrm{i}\boldsymbol{B}\rightarrow \mathrm{e}^{\mathrm{i}\theta}(\boldsymbol{E}+\mathrm{i}\boldsymbol{B})$$

which is not possible when the variables are the potentials. Since $^{**}F=-F$, (22) implies

$${}^{*}F_{\mu\nu}\rightarrow -F_{\mu\nu}\sin\theta+{}^{*}F_{\mu\nu}\cos\theta, \tag{23}$$

and in the absence of sources all components of $\mathscr{L}_{\alpha}$ are clearly invariant. (Note also the invariance of the conserved quantities (11)-(13), (15) and (16).) This gives rise to a set of four conserving currents forming the tensor

$$\begin{aligned} T^{\nu}_{\alpha}&=\frac{\partial\mathscr{L}_{\alpha}}{\partial(\partial_{\nu}F^{\rho\sigma})}\frac{\mathrm{d}}{\mathrm{d}\theta}(F_{\rho\sigma}\cos\theta+{}^{*}F_{\rho\sigma}\sin\theta)_{\theta=0}\\ &=2F_{\alpha\lambda}F^{\nu\lambda}-\tfrac{1}{2}F_{\kappa\lambda}F^{\kappa\lambda}\delta^{\nu}_{\alpha} \end{aligned} \tag{24}$$

which is the usual energy-momentum tensor. Thus, in this formulation, energy-momentum conservation is associated not with translational invariance but with invariance under duality rotations.

I am grateful to Dr N Anderson for a number of interesting conversations and for drawing my attention to the literature.

Note added in proof. Another connection between translational invariance and Lipkin's conserved tensor has been demonstrated by D B Fairlie (1965 *Nuovo Cimento* **37** 897).

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